

# Well-posedness of the non-local conservation law by stochastic perturbation.

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## Abstract

Stochastic non-local conservation law equation in the presence of discontinuous flux functions is considered in an  $L^1 \cap L^2$  setting. The flux function is assumed bounded and integrable (spatial variable). Our main result is to prove existence and uniqueness of weak solutions. The solution is strong solution in the probabilistic sense. The proof are constructive are based on the method of characteristics (in the presence of noise), Itô-Wentzell-Kunita formula and preceding of regularization. Our results are new, to the best of our knowledge, and are the first nonlinear extension of the seminar paper [20] where the linear case was addressed.

## 1 Introduction

We consider the conservation law

$$\begin{cases} \partial_t u(t, x) + \operatorname{Div}(F(t, x, u)) = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1.1)$$

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Here  $u$  is called the conserved quantity while  $F$  is the flux. These type of the equations express the balance equations of continuum physics, when small dissipation effects are neglected. A basic example is provided by the equations of gas and fluid dynamics, traffic flow and sedimentation of solid particles in a liquid. The well-posedness theorems within the class of entropy solutions were established by Kruzkov, see [24]. The selection of the physically relevant solution is based on the so called entropy condition that assert that a shock is formed only when the characteristics carry information toward the shock.

In 1995 was introduced by Lions, Perthame and Tadmor [28] the notions of called kinetic solution and relies on a new equation, the so-called kinetic formulation, that is derived from the conservation law at hand and that (unlike the original problem) possesses a very important feature - linearity. The two notions of solution, i.e. entropy and kinetic, are equivalent whenever both of them exist, nevertheless, kinetic solutions are more general as they are well defined even in situations when neither the original conservation law or the corresponding entropy inequalities can be understood in the sense of distributions. Among other significant references in this direction, let us emphasize the works of Chen and Perthame [9], Perthame [39] and P.L. Lions, P. Benoit and P.E. Souganidis [34].

Recently there has been an interest in studying the effect of stochastic forcing on nonlinear conservation laws [10, 14, 18, 23]. These papers consider the following stochastic scalar conservation laws

$$du(t, x) + \text{Div}(F(u))dt = g(u)dW(t, x),$$

with particular emphasis on existence and uniqueness questions (well-posedness).

For other hand, in [32] and [33] P.L. Lions, P. Benoit and P.E. Souganidis, introduced the theory of pathwise solutions to study the following stochastic conservation law

$$du(t, x) + \text{Div}(F(t, x, u)) \circ dz_t = 0,$$

where  $z_t$  is continuous noise. They defined a new concept of the solution and they proved existence and uniqueness via kinetic formulation. When the  $z_t$  is the standard Brownian motion the pathwise solutions corresponding formally on the Stratonovich interpretation of the equation. The theory is based on the general concepts/philosophy around the theory of stochastic

viscosity solutions for fully nonlinear first- and second-order PDE including stochastic Hamilton-Jacobi equations which was introduced and developed by Lions and Souganidis in [29, 30, 31].

The purpose of the present paper is a contribution to the following general question: can one hope for uniqueness of weak solutions of some class of conservation laws with irregular flux under stochastic perturbation. We present the first positive result. More precisely, we study the the following stochastic conservation law

$$\begin{cases} \partial_t u(t, x) + \text{Div}((F(t, x, (K * u)(x)) + \frac{dB_t}{dt}) \cdot u(t, x)) = 0, , \\ u|_{t=0} = u_0. \end{cases} \quad (1.2)$$

Here,  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $\omega \in \Omega$  is an element of the probability space  $(\Omega, \mathbb{P}, \mathcal{F})$ ,  $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $B_t$  is a standard Brownian motion and  $K$  is a regular kernel. The stochastic integration is to be understood in the Stratonovich sense. The Stratonovich form is the natural one for several reasons, including physical intuition related to the Wong-Zakai principle.

The novelty of our results is to show existence and uniqueness of the weak solutions for one-dimensional stochastic nonlocal conservation law (1.2) when  $F$  has low regularity in the spatial variable. Conservation laws with non-local fluxes have appeared recently in the literature, arising naturally in many fields of application, such as in crowd dynamics (see [6] and the references therein), or in models inspired from biology( see [5, 22]).

In recent years there has been an increasing interest in random influences on PDE. It can take into account for example, physical, empirical or numerical uncertainties and is commonly in physic models. The questions of regularizing effects and well-posedness by noise for partial differential equations have attracted much interest in recent years. The literature on regularization (i.e. improvement on uniqueness) by noise is vast and giving a complete survey at this point would exceed the purpose of this paper. Concerning the case of transport/continuity equations with irregular drift, we mention the works [3, 4, 16, 17, 20, 35, 36, 37, 38]. In particular, we would like to emphasize the papers [36] and [38] since the proofs have served as an inspiration for some steps of this work.

The existence proof is based in the method of characteristics, some estimation on the flow associated to characteristics, in the point fix arguments and stochastic calculus techniques.

We formally describe the proof of uniqueness :

- If  $U = U_1 - U_2$  is the different of two solutions of the stochastic conservation law (1.2), we have that  $U$  verifies

$$\begin{aligned} \partial_t U(t, x) + (F(t, x, U_1 * K)U + \frac{dB_t}{dt}) \cdot \nabla U(t, x) \\ = (F(t, x, U_2 * K) - F(t, x, U_1 * K))U_2. \end{aligned}$$

- If  $V$  is a primitive of  $U$  we have

$$\begin{aligned} \partial_t V(t, x) + (F(t, x, U_1 * K)V + \frac{dB_t}{dt}) \cdot \nabla V(t, x) \\ = (F(t, x, U_2 * K) - F(t, x, U_1 * K))U_2. \end{aligned}$$

- After we consider the characteristics associated to the flow (for  $F$  irregular in spatial variable)

$$dX_t = F(t, X_t, (U^1 * K)(t, X_t))dt + dB_t.$$

- By Itô formula we have

$$V(t, x)$$

$$= \int_0^t (F(s, (X_{t-s}^{-1}, (U_2 * K)(s, X_{t-s}^{-1})) - F(t, (X_{t-s}^{-1}, (U_2 * K)(s, X_{t-s}^{-1})))U_2(X_{t-s}^{-1})ds.$$

- Then using same estimation on the flow  $X_t$  we conclude

$$\int |V(t, x)|dx \leq C \int_0^t \int |V(t, x)|dx ds.$$

From this we deduce that  $V = 0$  and this implies that  $U = 0$

## 1.1 Hypothesis.

We denote  $F_1 = \partial_1 F$ ,  $F_3 = \partial_3 F$  and  $F_{3,3} = \partial_3^2 F$ . In this paper we assume the following hypothesis:

**Hypothesis 1.1.** *The flux  $F$  satisfies*

$$F \in L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R}))), \quad (1.3)$$

$$F \in L^\infty([0, T] \times \mathbb{R} \times \mathbb{R}), \quad (1.4)$$

$$F_1 \in L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R}))), \quad (1.5)$$

$$F_3 \in L^\infty([0, T] \times \mathbb{R} \times \mathbb{R}), \quad (1.6)$$

$$F_{3,3} \in L^2([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R}))), \quad (1.7)$$

Moreover, the initial condition verifies

$$u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \quad (1.8)$$

and the kernel satisfies

$$K \in C_0^\infty(\mathbb{R}), \quad (1.9)$$

## 1.2 Notations

First, through of this paper, we fix a stochastic basis with a  $d$ -dimensional Brownian motion  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$ . Then, we recall to help the intuition, the following definitions

$$\text{It\^o: } \int_0^t X_s dB_s = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} X_{t_i} (B_{t_{i+1} \wedge t} - B_{t_i}),$$

$$\text{Stratonovich: } \int_0^t X_s \circ dB_s = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} \frac{(X_{t_i \wedge t} + X_{t_i})}{2} (B_{t_{i+1} \wedge t} - B_{t_i}),$$

$$\text{Covariation: } [X, Y]_t = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} (X_{t_i \wedge t} - X_{t_i})(Y_{t_{i+1} \wedge t} - Y_{t_i}),$$

where  $\pi_n$  is a sequence of finite partitions of  $[0, T]$  with size  $|\pi_n| \rightarrow 0$  and elements  $0 = t_0 < t_1 < \dots$ . The limits are in probability, uniformly in time on compact intervals. Details about these facts can be found in Kunita [26]. Also we address from that book, Itô's formula, the chain rule for the stochastic integral, for any continuous d-dimensional semimartingale  $X = (X_1, X_2, \dots, X_d)$ , and twice continuously differentiable and real valued function  $f$  on  $\mathbb{R}^d$ .

## 2 Definition and Existence of Solutions.

### 2.1 $L^2$ - Solutions.

### 2.2 Definition of solutions

**Definition 2.1.** *A stochastic process  $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1(\Omega \times [0, T] \times \mathbb{R})$  is called a  $L^2$ - weak solution of the Cauchy problem (1.2) when: For any  $\varphi \in C_0^\infty(\mathbb{R})$ , the real valued process  $\int u(t, x)\varphi(x)dx$  has a continuous modification which is an  $\mathcal{F}_t$ -semimartingale, and for all  $t \in [0, T]$ , we have  $\mathbb{P}$ -almost surely*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}} u_0(x)\varphi(x) dx \\ &+ \int_0^t \int_{\mathbb{R}} u(s, x) F(s, x, (K * u)) \partial_x \varphi(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx \circ dB_s. \end{aligned} \quad (2.10)$$

**Remark 2.2.** *Using the same idea as in Lemma 13 [20], one can write the problem (1.2) in Itô form as follows, a stochastic process  $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1(\Omega \times [0, T] \times \mathbb{R})$  is a  $L^2$ - weak solution of the SPDE (1.2) iff for every test function  $\varphi \in C_0^\infty(\mathbb{R})$ , the process  $\int u(t, x)\varphi(x)dx$  has a continuous modification which is a  $\mathcal{F}_t$ -semimartingale and satisfies the following Itô's formulation*

$$\int_{\mathbb{R}} u(t, x)\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) F(s, x, (K * u)) \partial_x \varphi(x) dx ds$$

$$+ \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds.$$

### 2.3 Existence.

The goal of this section is to prove general existence result for stochastic conservation law with low regularity of the flux function.

**Lemma 2.3.** *Assume that hypothesis 1.1 holds. Then there exist  $L^2$ -weak solutions of the Cauchy problem (1.2).*

*Proof. Step 1: Regularization.*

Let  $\{\rho_n\}_n$  be a family of standard symmetric mollifiers, We define the family of regularized coefficients given by

$$F^n(t, \cdot, z) = (F *_{x} \rho_n)(t, \cdot, z)$$

and

$$u_0^n = u_0 * \rho_n.$$

Clearly we observe that, for every  $n \in \mathbb{N}$ , any element  $F^n$ ,  $u_0^n$  are smooth with bounded derivatives of all orders. We define  $u^1$  as the solution of the following SPDE

$$\begin{cases} du^1(t, x) + Div \left( u^1(t, x) \cdot (F^1(t, x, (u_0^1 * K))dt + \circ dB_t) \right) = 0, \\ u^\varepsilon|_{t=0} = u_0^1. \end{cases} \quad (2.11)$$

Moreover, inductively we define

$$\begin{cases} du^{n+1}(t, x) + Div \left( u^{n+1}(t, x) \cdot (F^{n+1}(t, x, (u^n * K))dt + \circ dB_t) \right) = 0, \\ u^{n+1}|_{t=0} = u_0^{n+1}. \end{cases} \quad (2.12)$$

Following the classical theory of H. Kunita [25, Theorem 6.1.9] we obtain that

$$u^{n+1}(t, x) = u_0^{n+1}(\psi_t^{n+1}(x))J\psi_t^{n+1}(x)$$

is the unique solution to the regularized equation (2.12), where  $\phi_t^{n+1}$  is the flow associated to the following stochastic differential equation (SDE):

$$dX_t = F^{n+1}(s, X_s, (u^n * K)(s, X_s)) dt + dB_t, \quad X_0 = x,$$

and  $\psi_t^{n+1}$  is the inverse of  $\phi_t^{n+1}$ .

*Step 2: Boundedness.* We postpone the following estimation for the appendix

$$\mathbb{E} \left[ \left| \partial_x X_{s,t}^n(x) \right|^{-1} \right] \leq C. \quad (2.13)$$

Then making the change of variables  $y = \psi_t^\varepsilon(x)$  we have that

$$\int_{\mathbb{R}} \mathbb{E}[|u^{n+1}(t, x)|^2] dx = \int_{\Omega} \int_{\mathbb{R}} |u_0^{n+1}(y)|^2 (J\phi_t^{n+1}(x))^{-1} dx.$$

Now, by inequality (2.13) we conclude

$$\int_{\mathbb{R}} \mathbb{E}[|u^n(t, x)|^2] dx \leq C \int_{\mathbb{R}} |u_0^n(y)|^2 dx. \quad (2.14)$$

Therefore, the sequence  $\{u^n\}$  is bounded in  $u \in L^2(\Omega \times [0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$ . Then there exists a convergent subsequence, which we denote also by  $u^n$ , such that converge weakly in  $L^2(\Omega \times [0, T] \times \mathbb{R})$  and weak-star in  $L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$  to some process  $u \in L^2(\Omega \times [0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$ . Since this subsequence is bounded in  $L^1(\Omega \times [0, T] \times \mathbb{R})$  we follows that  $u^n$  converge to the measure  $\mu$  and  $\mu = u$ .

*Step 3: Passing to the Limit.* Now, if  $u^{n+1}$  is a solution of (2.12), it is also a weak solution, that is, for any test function  $\varphi \in C_0^\infty(\mathbb{R})$ ,  $u^{n+1}$  verifies (written in the Itô form):

$$\begin{aligned} \int_{\mathbb{R}} u^{n+1}(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0^{n+1}(x) \varphi(x) dx \\ &+ \int_0^t \int_{\mathbb{R}} u^{n+1}(s, x) F^{n+1}(s, x, (u^n * K)) \partial_x \varphi(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} u^{n+1}(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^{n+1}(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned} \quad (2.15)$$



Now, we observe that  $G^{n+1} = u^{n+1}(s, x) F^{n+1}(s, x, u^n * K)$  is uniformly bounded in  $L^2(\Omega \times [0, T] \times \mathbb{R})$ . Then there exists a convergent subsequence, which we denote also by  $G^n$ , such that converge weakly in  $L^2(\Omega \times [0, T] \times \mathbb{R})$  to some process  $G \in L^2(\Omega \times [0, T] \times \mathbb{R})$ .

Then passing to the limit in equation (2.15) along the convergent subsequences found, we have

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} G(s, x, \omega) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

From the last equality we have that  $\int_{\mathbb{R}} u(t, x) \varphi(x) dx$  is continuous semimartingale for any test function  $\varphi \in C_0^\infty(\mathbb{R})$ . Thus  $u * K$  is continuous semimartingale. We observe that  $u^{n+1} * K$  converge to  $u * K$  and that  $F^{n+1}(s, x, (u^n * K))$  strong converge to  $F(s, x, (u * K))$  in  $L^2([0, T] \times \Omega, L_{loc}^2(\mathbb{R}))$ . Then passing to the limit in equation (2.15) along the convergent subsequences found, we conclude that

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) F(s, x, (u * K)) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

□

### 3 Uniqueness.

#### 3.1 Estimation on the flow.

Assume that  $u$  is a  $L^2$ -solution of the Cauchy problem (1.2), . Let  $\{\rho_\varepsilon\}_\varepsilon$  be a family of standard symmetric mollifiers. We denoted  $F_\varepsilon(t, \cdot, (u * K)) = (F * \rho_\varepsilon)(t, \cdot, (u * K))$ . We observ that the regularization is done in the second variable

**Lemma 3.1.** *Assume the hypothesis 1.1. Then for  $T > 0$  there exist a constant  $C$  such that*

$$\mathbb{E} \left[ \left| \partial_x X_{s,t}^\varepsilon(x) \right|^{-1} \right] \leq C, \quad (3.16)$$

where  $C$  depend on  $\|F\|_{L^\infty([0,T],L^1(\mathbb{R},L^\infty(\mathbb{R})))}$ ,  $\|F\|_{L^\infty([0,T]\times\mathbb{R}\times\mathbb{R})}$ ,  $\|F_1\|_{L^\infty([0,T],L^1(\mathbb{R},L^\infty(\mathbb{R})))}$ ,  $\|F_3\|_{L^\infty([0,T]\times\mathbb{R}\times\mathbb{R})}$  and  $\|F_{3,3}\|_{L^2([0,T],L^1(\mathbb{R},L^\infty(\mathbb{R})))}$ .

*Proof. Step 1: Regularization.*

We consider the SDE associated to the vector field  $b$  :

$$dX_t^\varepsilon = F_\varepsilon(t, X_t, (u * K)(t, X_t)) dt + dB_t, \quad X_0 = x.$$

We note that  $\partial_x X_{s,t}$  satisfies

$$\partial_x X_t^\varepsilon = \exp \left\{ \int_0^t \partial_x (F_\varepsilon(s, x, (u * K))) (s, X_s) ds \right\}.$$

*Step 2: Semimartingale representation.* By definition of solution we have that  $(K * u)(t, x)$  is a continuous semimartingale and verifies

$$\begin{aligned} \int_{\mathbb{R}} u(t, y) K(x - y) dy &= \int_{\mathbb{R}} u_0(y) K(x - y) dy \\ + \int_0^t \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) dy ds &+ \int_0^t \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dx dB_s \\ + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, y) \partial_y^2 K(x - y) dy ds. \end{aligned}$$

Then by Itô formula we have

$$\begin{aligned} F(t, x, (K * u)(x)) &= F(0, x, (K * u_0)(x)) + \int_0^t F_1(s, x, (K * u)) ds \\ + \int_0^t F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) dy ds \\ + \int_0^t F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy \circ dB_s \end{aligned} \quad (3.17)$$

In order to write the last equality in Itô formulation we have to calculate the covariation

$$[\partial_3 F(s, x, (K * u)) \int_{\mathbb{R}} u(s, x) \partial_y K(x - y) dy, B_s].$$

We observe that

$$\begin{aligned}
F_3(t, x, (K * u)(x)) &= F_3(0, x, (K * u_0)(x)) + \int_0^t F_{1,3}(s, x, (K * u)(x)) \, ds \\
&+ \int_0^t F_{3,3}(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) \, dy \, ds \\
&+ \int_0^t F_{3,3}(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) \, dy \circ dB_s
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} u(t, x) \partial_y K(x - y) \, dy &= \int_{\mathbb{R}} u_0(y) \partial_y K(x - y) \, dy \\
&+ \int_0^t \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y^2 K(x - y) \, dy \, ds \\
&+ \int_0^t \int_{\mathbb{R}} u(s, y) \partial_y^2 K(x - y) \, dy \circ dB_s
\end{aligned}$$

We set  $u^k(t, x) := (K * u)(t, x)$ . From the Itô formula for the product of two semimartingales we obtain

$$\begin{aligned}
F_3(t, x, (K * u)(x)) \partial_x u^k(t, x) &= F_3(0, x, (K * u_0)(x)) \partial_x u_0^k(t, x) \\
&+ \int_0^t \partial_x u^k(s, x) F_{3,1}(s, x, (K * u)) \, ds \\
&+ \int_0^t \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \int_{\mathbb{R}} u(s, x) F(s, x, (K * u)) \partial_x K(y - x) \, dx \, ds \\
&+ \int_0^t \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) \circ dB_s \\
&+ \int_0^t F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y^2 K(x - y) \, dy \, ds \\
&+ \int_0^t F_3(s, x, (K * u)) \partial_x^2 u^k(s, x) \circ dB_s
\end{aligned}$$

Applying covariation in the last equality we conclude that

$$\begin{aligned}
& [\partial_3 F(s, x, (K * u)) \int_{\mathbb{R}} u(s, x) \partial_y K(x - y) dy, B_s] \\
&= \int_0^t \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) ds \\
&\quad + \int_0^t F_3(s, x, (K * u)) \partial_x^2 u^k(s, x) ds. \tag{3.18}
\end{aligned}$$

From formulas (3.17) and (3.18) we deduce

$$\begin{aligned}
F(t, x, (K * u)(x)) &= F(0, x, (K * u_0)(x)) + \int_0^t F_1(s, x, (K * u)) ds \\
&+ \int_0^t F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) dy ds \\
&\quad + \int_0^t F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dB_s \\
&\quad + \int_0^t \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) ds \\
&\quad + \int_0^t F_3(s, x, (K * u)) \partial_x^2 u^k(s, x) ds.
\end{aligned}$$

Integrating the last equality we have

$$\begin{aligned}
& \int_{-\infty}^z F(t, x, (K * u)) dx = \int_{-\infty}^z F(0, x, (K * u_0)) dz \\
&\quad + \int_0^t \int_{-\infty}^z F_1(s, x, (K * u)) dx ds \\
&+ \int_0^t \int_{-\infty}^z F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) dy dx ds \\
&\quad + \int_0^t \int_{-\infty}^z F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \\
&\quad + \int_0^t \int_{-\infty}^z \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) dx ds
\end{aligned}$$

$$+ \int_0^t \int_{-\infty}^z F_3(s, x, (K * u)) \partial_x^2 u^k(s, x) dx ds.$$

*Step 3: Itô-Wentzell-Kunita formula.*

Now, we applying the Itô-Wentzell-Kunita formula to  $\int_{-\infty}^{X_t} F(t, x, (K * u)(x)) dx$ , see Theorem 8.3 of [26], we have

$$\begin{aligned} \int_{-\infty}^{X_t} F(t, x, (K * u)) dx &= \int_{-\infty}^x F(0, x, (K * u_0)) dz \\ &+ \int_0^t \int_{-\infty}^{X_s} F_1(s, x, (K * u)) dx ds \\ &+ \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) dy dx ds \\ &+ \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \\ &+ \int_0^t \int_{-\infty}^{X_s} \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) dx ds \\ &+ \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \partial_x^2 u^k(s, x) dx ds \\ &+ \int_0^t F^2(s, X_s, (K * u)(s, X_s)) ds + \int_0^t F(s, X_s, (K * u)(s, X_s)) dB_s \\ &+ \frac{1}{2} \int_0^t \partial_x (F(s, x, (K * u)))(s, X_s) ds + \int_0^t F_3(s, X_s, (K * u)) \partial_x u^k(s, X_s) ds \end{aligned}$$

*Step 4: Boundedness.*

We observe that

$$\left\| \int_{-\infty}^{X_t} F(t, x, (K * u)) dx \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq \|F\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_{-\infty}^x F(0, x, (K * u_0)) dx \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq C \|F\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_1(s, x, (K * u)) \, dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq C \|F_1\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u)) \partial_y K(x - y) \, dy dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|F\|_{L^\infty([0, T] \times \Omega \times \mathbb{R})} \|u\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))} \|F_3\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \left| \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) \, dy dx \right|^2 ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|u\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_3\|_{L^2([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} \partial_x u^k(s, x) F_{3,3}(s, x, (K * u)) \partial_x u^k(s, x) \, dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|u\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_{3,3}\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \partial_{x,x} u^k(s, x) \, dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|u\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_3\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))},$$

$$\left\| \int_0^t F^2(s, X_s, (K * u)(s, X_s)) ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq \|F\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})}^2,$$

$$\left\| \int_0^t F_3(s, X_s, (K * u)(s, X_s)) \partial_x u^k(s, X_s) ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq \|u\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))} \|F_3\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})}^2.$$

*Step 5: Conclusion.*

We set

$$\begin{aligned} & \mathcal{E} \left( \int_0^t F(s, X_s, (K * u)(s, X_s)) dB_s \right) \\ &= \exp \left\{ \int_0^t F^2(s, X_s, (K * u)(s, X_s)) dB_s - \frac{1}{2} \int_s^t F^2(s, X_s, (K * u)(s, X_s)) ds \right\}, \end{aligned}$$

and

$$\begin{aligned} & \mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \right) \\ &= \exp \left\{ \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \right. \\ & \quad \left. - \frac{1}{2} \int_s^t \left| \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx \right|^2 ds \right\}. \end{aligned}$$

From inequalities in step 4 and Hölder inequality we deduce that

$$\begin{aligned} \mathbb{E} \left[ \left| \frac{dX_{s,t}}{dx}(x) \right|^{-1} \right] &\leq \\ &\leq C \mathcal{E} \left( \int_0^t F(s, X_s, (K * u)(s, X_s)) dB_s \right) \\ &\times \mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \right) \end{aligned} \quad (3.19)$$

Finally we observe that the processes

$$\mathcal{E} \left( \int_0^t F(s, X_s, (K * u)(s, X_s)) dB_s \right)$$

and

$$\mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u)) \int_{\mathbb{R}} u(s, y) \partial_y K(x - y) dy dx dB_s \right)$$

are martingales with expectation equal to one. From this we conclude our lemma.  $\square$

**Remark 3.2.** *The same results is valid for the backward flow  $Y_{s,t}$  since it is solution of the same SDE driven by the drifts  $-b$ .*

### 3.2 Main result.

In this section, we shall present a uniqueness theorem for the SPDE (1.2). The proof is based on applying the characteristic method, the commutator Lemma preceding regularization to a primitive of the solution. We pointed that similar arguments were used in previous work [38] for linear continuity equation.

**Theorem 3.3.** *Under the conditions of hypothesis 1.1, uniqueness holds for  $L^2$ -weak solutions of the Cauchy problem (1.2) in the following sense: if  $u, v$  are  $L^2$ -weak solutions with the same initial data  $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ , then  $u = v$  almost everywhere in  $\Omega \times [0, T] \times \mathbb{R}$ .*

*Proof. Step 1: Two solutions.* Let  $v$  and  $w$  be two  $L^2$ -weak solutions with initial conditions equal to  $u_0$ . We denote  $u = v - w$ . Then  $u$  verifies

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx = & \\ & + \int_0^t \int_{\mathbb{R}} u(s, x) F(s, x, (K * v)) \partial_x \varphi(x) dx ds \\ & + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx \circ dB_s \\ & + \int_0^t \int_{\mathbb{R}} w(s, x) (F(s, x, (K * w)) - F(s, x, (K * v))) \partial_x \varphi(x) dx ds \end{aligned}$$

*Step 2: Primitive of the solution.* We set

$$V(t, x) = \int_{-\infty}^x u(t, y) dy.$$

We observe that  $\partial_x V(t, x) = u(t, x)$  belong to  $L^2(\Omega \times [0, T] \times \mathbb{R})$ . Now, we consider a nonnegative smooth cut-off function  $\eta$  supported on the ball of radius 2 and such that  $\eta = 1$  on the ball of radius 1. For any  $R > 0$ , we introduce the rescaled functions  $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$ .

For all test functions  $\varphi \in C_0^\infty(\mathbb{R})$  we obtain

$$\int_{\mathbb{R}} V(t, x) \varphi(x) \eta_R(x) dx = - \int_{\mathbb{R}} u(t, x) \theta(x) \eta_R(x) dx - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx,$$



where  $\theta(x) = \int_{-\infty}^x \varphi(y) dy$ . By definition of  $L^2$ -solutions, taking as test function  $\theta(x)\eta_R(x)$  we get

$$\begin{aligned}
& \int_{\mathbb{R}} V(t, x) \eta_R(x) \varphi(x) dx = \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) F(s, x, (K * v)) \eta_R(x) \varphi(x) dx ds \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \eta_R(x) \varphi(x) dx \circ dB_s \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) F(s, x, (K * v)) \partial_x \eta_R(x) \theta(x) dx ds \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \circ dB_s - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx \\
& - \int_0^t \int_{\mathbb{R}} w(s, x) (F(s, x, (K * w)) - F(s, x, (K * v))) \eta_R(x) \varphi(x) dx ds \\
& - \int_0^t \int_{\mathbb{R}} w(s, x) (F(s, x, (K * w)) - F(s, x, (K * v))) \partial_x \eta_R(x) \varphi(x) dx ds
\end{aligned} \tag{3.20}$$

We observe that

$$\int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \circ dB_s \rightarrow 0,$$

$$\int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx \rightarrow 0,$$

$$\int_0^t \int_{\mathbb{R}} w(s, x) (F(s, x, (K * w)) - F(s, x, (K * v))) \partial_x \eta_R(x) \varphi(x) dx ds \rightarrow 0,$$

$$\int_0^t \int_{\mathbb{R}} \partial_x V(s, x) F(s, x, (K * v)) \partial_x \eta_R(x) \theta(x) dx ds \rightarrow 0$$

as  $R \rightarrow \infty$ . Passing to the limit in equation (3.20) we have that

$$\begin{aligned}
& \int_{\mathbb{R}} V(t, x) \varphi(x) dx = \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) F(s, x, (K * v)) \varphi(x) dx ds \\
& - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \varphi(x) dx \circ dB_s \\
& - \int_0^t \int_{\mathbb{R}} w(s, x) (F(s, x, (K * w)) - F(s, x, (K * v))) \varphi(x) dx ds
\end{aligned}$$

*Step 3: Smoothing.* Let  $\{\rho_\varepsilon(x)\}_\varepsilon$  be a family of standard symmetric mollifiers. For any  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$  we use  $\rho_\varepsilon(x - \cdot)$  as test function, then we deduce

$$\begin{aligned}
& \int_{\mathbb{R}} V(t, y) \rho_\varepsilon(x - y) dy = \\
& - \int_0^t \int_{\mathbb{R}} (F(s, y, (K * v)) \partial_y V(s, y)) \rho_\varepsilon(x - y) dy ds \\
& - \int_0^t \int_{\mathbb{R}} \partial_y V(s, y) \rho_\varepsilon(x - y) dy \circ dB_s \\
& - \int_0^t \int_{\mathbb{R}} w(s, y) (F(s, y, (K * w)) - F(s, y, (K * v))) \rho_\varepsilon(x - y) dy ds
\end{aligned}$$

We denote  $V_\varepsilon(t, x) = (V * \rho_\varepsilon)(x)$ ,  $F_\varepsilon(s, x, (K * v)(x)) = (F * \rho_\varepsilon)(x)$  and  $(FV)_\varepsilon(t, x) = (F.V * \rho_\varepsilon)(x)$ . Thus we have

$$\begin{aligned}
& V_\varepsilon(t, x) + \int_0^t F_\varepsilon(s, x, \omega) \partial_x V_\varepsilon(s, x) ds + \int_0^t \partial_x V_\varepsilon(s, x) \circ dB_s \\
& = \int_0^t (\mathcal{R}_\varepsilon(V, F))(x, s) ds \\
& - \int_0^t \int_{\mathbb{R}} w(s, x) (F(s, y, (K * w)) - F(s, y, (K * v))) \rho_\varepsilon(x - y) dy ds
\end{aligned}$$

where we denote  $\mathcal{R}_\varepsilon(V, F) = F_\varepsilon \partial_x V_\varepsilon - (F \partial_x V)_\varepsilon$ .

*Step 4: Method of Characteristics.* We consider the flow

$$dX_t = F^\varepsilon(s, X_t) dt + dB_t, \quad X_0 = x,$$

Applying the Itô-Wentzell-Kunita formula to  $V_\varepsilon(t, X_t^\varepsilon)$ , see Theorem 8.3 of [26], we have

$$\begin{aligned} V_\varepsilon(t, X_t^\varepsilon) &= \int_0^t (\mathcal{R}_\varepsilon(V, F))(X_s^\varepsilon, s) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} w(F(s, x, (K * w)) - F(s, x, (K * v))) \rho_\varepsilon(X_s^\varepsilon - x) dx ds \end{aligned}$$

Then, considering that  $X_t^\varepsilon = X_{0,t}^\varepsilon$  and  $Y_t^\varepsilon = Y_{0,t}^\varepsilon = (X_{0,t}^\varepsilon)^{-1}$  we deduce that

$$\begin{aligned} V_\varepsilon(t, x) &= \int_0^t (\mathcal{R}_\varepsilon(V, F))(Y_{t-s}^\varepsilon, s) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} w(F(s, x, (K * w)) - F(s, x, (K * v))) \rho_\varepsilon(Y_{t-s}^\varepsilon - y) dy ds \end{aligned}$$

*Step 5: Localization.* Now, we consider a nonnegative smooth cut-off function  $\eta$  supported on the ball of radius 2 and such that  $\eta = 1$  on the ball of radius 1. For any  $R > 0$ , we introduce the rescaled functions  $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$ . From the last step we have

$$\begin{aligned} &\int_{\mathbb{R}} |V_\varepsilon(t, x)| \eta_R(x) dx \\ &\leq \int_0^t \int_{\mathbb{R}} |(\mathcal{R}_\varepsilon(V, F))(Y_{t-s}^\varepsilon, s)| \eta_R(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \eta_R(x) \int_{\mathbb{R}} |w| |F(s, y, (K * w)) - F(s, y, (K * v))| \rho_\varepsilon(Y_{t-s}^\varepsilon - y) dy dx ds \end{aligned} \tag{3.21}$$

*Step 6: Convergence of the commutator I.* Now, we observe that  $\mathcal{R}_\varepsilon(V, b)$  converge to zero in  $L^2([0, T] \times \Omega \times \mathbb{R})$ . In fact, we get that

$$(F \partial_x V)_\varepsilon \rightarrow F \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

Moreover, we have

$$F_\varepsilon \rightarrow F \text{ in } L^1([0, T] \times \Omega \times \mathbb{R})$$

and

$$\partial_x V_\varepsilon \rightarrow \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

Then by the dominated convergence theorem we obtain

$$F_\varepsilon \partial_x V_\varepsilon \rightarrow F \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

*Step 7: Convergence of the commutator II.* We observe that

$$\begin{aligned} & \int_0^t \int |(\mathcal{R}_\varepsilon(V, F))(Y_{t-s}^\varepsilon, s)| \eta_R(x) \, dx \, ds \\ &= \int_0^t \int |(\mathcal{R}_\varepsilon(V, F))(s, x)| JX_{t-s}^\varepsilon \eta_R(X_{t-s}^\varepsilon) \, dx \, ds. \end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \int (\mathcal{R}_\varepsilon(V, F))(x, s) JX_{s,t}^\varepsilon \eta_R(X_{t-s}^\varepsilon) \, dx \, ds \right| \\ & \leq \left( \mathbb{E} \int_0^t \int |(\mathcal{R}_\varepsilon(V, F))(x, s)|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^t \int |JX_{s,t}^\varepsilon \eta_R(X_{t-s}^\varepsilon)|^2 \, dx \, ds \right)^{\frac{1}{2}} \end{aligned}$$

From step 6 we follow

$$\left( \mathbb{E} \int_0^t \int |(\mathcal{R}_\varepsilon(V, F))(x, s)|^2 \, dx \, ds \right)^{\frac{1}{2}} \rightarrow 0.$$

From lemma 3.1 we deduce

$$\left( \mathbb{E} \int_0^t \int |JX_{s,t}^\varepsilon \eta_R(X_{t-s}^\varepsilon)|^2 \, dx \, ds \right)^{\frac{1}{2}} = \left( \mathbb{E} \int_0^t \int |JY_{s,t}^\varepsilon|^{-1} |\eta_R(x)|^2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$\leq C \left( \int |\eta_R(x)|^2 dx \right)^{\frac{1}{2}},$$

*Step 8: Conclusion .* From step 5 we have

$$\begin{aligned} & \int_{\mathbb{R}} |V_\varepsilon(t, x)| \eta_R(x) dx \\ & \leq \int_0^t \int_{\mathbb{R}} |(\mathcal{R}_\varepsilon(V, F))(Y_{t-s}^\varepsilon, s)| \eta_R(x) dx ds \\ & + \int_0^t \int_{\mathbb{R}} \eta_R(x) \int_{\mathbb{R}} |w(s, x)| |F(s, y, (K*w)) - F(s, y, (K*v))| \rho_\varepsilon(Y_{t-s}^\varepsilon - y) dy dx ds ds \end{aligned}$$

Now we observe that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \eta_R(x) \int_{\mathbb{R}} |w(s, x)| |F(s, x, (K*w)) - F(s, x, (K*v))| \rho_\varepsilon(Y_{t-s}^\varepsilon - y) dy dx ds ds \\ & \leq C \|w\|_{L^1(\mathbb{R})} \int_0^t \int_{\tilde{K}} |V(s, z)| dz ds \end{aligned}$$

where  $\tilde{K}$  is the support of  $K$ . If we take  $R$  sufficiently large we have

$$\begin{aligned} & \int_{\mathbb{R}} |V_\varepsilon(t, x)| \eta_R(x) dx \\ & \leq \int_0^t \int_{\mathbb{R}} |(\mathcal{R}_\varepsilon(V, F))(Y_{t-s}^\varepsilon, s)| \eta_R(x) dx ds \\ & + C \|w\|_{L^1(\mathbb{R})} \int_0^t \int \eta_R(x) |V(s, z)| dz ds \end{aligned}$$

Taking the limit as  $\varepsilon$  converge to zero we deduce

$$\begin{aligned} & \int_{\mathbb{R}} |V(t, x)| \eta_R(x) dx \\ & \leq C \|w\|_{L^1(\mathbb{R})} \int_0^t \int \eta_R(x) |V(s, z)| dz ds \end{aligned}$$

By Gronwall lemma we deduce that  $V = 0$ . Then we have  $u = 0$ .

□

## 4 Appendix

In this lemma we use the arguments that in Lemma 3.1

**Lemma 4.1.** *Assume the hypothesis 1.1. Then for  $T > 0$  there exist a constant  $C$  such that*

$$\mathbb{E} \left[ \left| \partial_x X_{s,t}^n(x) \right|^{-1} \right] \leq C, \quad (4.22)$$

where  $C$  depends on  $\|F\|_{L^\infty([0,T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))}$ ,  $\|F\|_{L^\infty([0,T] \times \mathbb{R} \times \mathbb{R})}$ ,  $\|F_1\|_{L^\infty([0,T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))}$ ,  $\|F_3\|_{L^\infty([0,T] \times \mathbb{R} \times \mathbb{R})}$  and  $\|F_{3,3}\|_{L^2([0,T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))}$ .

*Proof. Step 1: Regularization.*

We consider the SDE :

$$dX_t^{n+1} = F_{n+1}(t, X_t, (u^n * K)(t, X_t)) dt + dB_t, \quad X_0 = x.$$

We observe that  $\partial_x X_t$  verifies

$$\partial_x X_t^{n+1} = \exp \left\{ \int_0^t \partial_x (F_n(s, x, (u^n * K)))(s, X_t) ds \right\}.$$

*Step 2: Semimartingale representation.* By definition of solution we get that  $(K * u)(t, x)$  satisfies

$$\begin{aligned} & \int_{\mathbb{R}} u^{n+1}(t, y) K(x - y) dy = \int_{\mathbb{R}} u_0(y) K(x - y) dy \\ & + \int_0^t \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (u^n * K)) \partial_y K(x - y) dy ds + \int_0^t \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dB_s \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y^2 K(x - y) dy ds. \end{aligned}$$

Then by Itô formula we have

$$\begin{aligned} F(t, x, (K * u^{n+1})) &= F(0, x, (K * u_0)) + \int_0^t F_1(s, x, (u^{n+1} * K)) ds \\ &+ \int_0^t F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y K(x - y) dy ds \end{aligned}$$

$$+ \int_0^t F(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy \circ dB_s \quad (4.23)$$

In order to write the last equality in Itô formulation we have to calculate the covariation

$$[F_3(s, x, (K * u^{n+1}))(x) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy, B_s].$$

Now, we obtain

$$\begin{aligned} F_3(t, x, (K * u^{n+1})) &= F_3(0, x, (K * u_0)) + \int_0^t F_{3,1}(s, x, (K * u^{n+1})) ds \\ &+ \int_0^t F_{3,3}(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u(s, y) F(s, y, (K * u^n)) \partial_y K(x - y) dy ds \\ &+ \int_0^t F_{3,3}(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy \circ dB_s \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} u^{n+1}(t, x) \partial_y K(x - y) dy &= \int_{\mathbb{R}} u_0(y) \partial_y K(x - y) dy \\ &+ \int_0^t \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_{y,y} K(x - y) dy ds \\ &+ \int_0^t \int_{\mathbb{R}} u^{n+1}(s, y) \partial_{y,y} K(x - y) dy \circ dB_s \end{aligned}$$

We set  $u^{n,k}(t, x) := (K * u^n)(t, x)$ . From the Itô formula for the product of two semimartingales we obtain

$$\begin{aligned} F_3(t, x, (K * u^{n+1})) \partial_x u^{n+1,k}(t, x) &= F_3(0, x, (K * u_0)) \partial_x u_0^k(t, x) \\ &+ \int_0^t \partial_x u^{n+1,k}(s, x) F_{3,1}(s, x, (K * u^{n+1})) ds \\ &+ \int_0^t \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u(s, x) F(s, x, (K * u^n)) \partial_x K(y - x) dx ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \partial_x u^{n+1,k}(s, x) \circ dB_s \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y^2 K(x - y) dy ds \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \partial_x^2 u^{n+1,k}(s, x) \circ dB_s
\end{aligned}$$

Applying covariation in the last equality we deduce

$$\begin{aligned}
& [F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, x) \partial_y K(x - y) dy, B_s] \\
& = \int_0^t \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \partial_x u^k(s, x) ds \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \partial_{x,x} u^k(s, x) ds. \tag{4.24}
\end{aligned}$$

From formulas (4.23) and (4.24) we obtain

$$\begin{aligned}
F(t, x, (K * u^{n+1})) & = F(0, x, (K * u_0)) + \int_0^t F_1(s, x, (K * u^{n+1})) ds \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y K(x - y) dy ds \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dB_s \\
& + \int_0^t \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \partial_x u^{n+1,k}(s, x) ds \\
& + \int_0^t F_3(s, x, (K * u^{n+1})) \partial_x^2 u^{n+1,k}(s, x) ds.
\end{aligned}$$

Integrating the last equality we get

$$\int_{-\infty}^z F(t, x, (K * u^{n+1})) dx = \int_{-\infty}^z F(0, x, (K * u_0)) dz$$



$$\begin{aligned}
& + \int_0^t \int_{-\infty}^z F_1(s, x, (K * u^{n+1})) \, dx ds \\
& + \int_0^t \int_{-\infty}^z F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y K(x-y) \, dy dx ds \\
& + \int_0^t \int_{-\infty}^z F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x-y) \, dy dx dB_s \\
& + \int_0^t \int_{-\infty}^z \partial_x u^{n+1,k}(s, x) \partial_{3,3} F(s, x, (K * u^{n+1})) \partial_x u^{n+1,k}(s, x) \, dx ds \\
& + \int_0^t \int_{-\infty}^z F_3(s, x, (K * u^{n+1})) \partial_x^2 u^{n+1,k}(s, x) \, dx ds.
\end{aligned}$$

*Step 3: Itô-Wentzell-Kunita formula.* Applying the Itô-Wentzell-Kunita formula to  $\int_{-\infty}^{X_t} F(t, x, (K * u)(x)) dx$ , see Theorem 8.3 of [26], we deduce

$$\begin{aligned}
& + \int_{-\infty}^{X_t} F(t, x, (K * u^{n+1})) dx = \int_{-\infty}^x F(0, x, (K * u_0)) dz \\
& + \int_0^t \int_{-\infty}^{X_s} F_1(s, x, (K * u^{n+1})) \, dx ds \\
& + \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y K(x-y) \, dy dx ds \\
& + \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x-y) \, dy dx dB_s \\
& + \int_0^t \int_{-\infty}^{X_s} \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \partial_x u^{n+1,k}(s, x) \, dx ds \\
& + \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \partial_{x,x} u^{n+1,k}(s, x) \, dx ds \\
& + \int_0^t F(s, X_s, (K * u^n)(s, X_s)) F(s, X_s, (K * u^{n+1})(s, X_s)) ds
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t F(s, X_s, (K * u^{n+1})(s, X_s)) dB_s \\
& + \frac{1}{2} \int_0^t (\partial_x F(s, x, (K * u^{n+1}))) (s, X_s) ds \\
& + \int_0^t F_3(s, X_s, (K * u^{n+1})) \partial_x u^{n+1,k}(s, X_s) ds
\end{aligned}$$

*Step 4: Boundedness.*

We have that

$$\left\| \int_{-\infty}^{X_t} F(t, x, (K * u^{n+1})) dx \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq \|F\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_{-\infty}^x F(0, x, (K * u_0)) dx \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq C \|F\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_1(s, x, (K * u^{n+1})) dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq C \|F_1\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_2(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) F(s, y, (K * u^n)) \partial_y K(x-y) dy dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|F\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})} \|u^{n+1}\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))} \|F_3\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\leq C \|F\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})} \|F_3\|_{L^\infty([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \left| \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})(x)) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x-y) dy dx \right|^2 ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} \leq C$$

$$\leq C \|u^{n+1}\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_3\|_{L^2([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\leq C \|F_3\|_{L^2([0, T], L^1(\mathbb{R}, L^\infty(\mathbb{R})))},$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} \partial_x u^{n+1,k}(s, x) F_{3,3}(s, x, (K * u^{n+1})) \partial_x u^{n+1,k}(s, x) dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\begin{aligned} &\leq C \|u^{n+1}\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_{3,3}\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))}, \\ &\leq C \|F_{3,3}\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))}, \end{aligned}$$

$$\left\| \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \partial_{x,x} u^{n+1,k}(s, x) dx ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq C \|u^{n+1}\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))}^2 \|F_3\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))},$$

$$\leq C \|F_3\|_{L^1([0, T] \times \mathbb{R}, L^\infty(\mathbb{R}))},$$

$$\left\| \int_0^t F(s, X_s, (K * u^{n+1})(s, X_s)) F(s, X_s, (K * u^n)(s, X_s)) ds \right\|_\infty \leq \|F\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})}^2,$$

$$\left\| \int_0^t F_3(s, X_s, (K * u^{n+1})(s, X_s)) \partial_x u^k(s, X_s) ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})}$$

$$\leq \|u^{n+1}\|_{L^\infty([0, T] \times \Omega, L^1(\mathbb{R}))} \|F_3\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})}.$$

$$\leq \|F_3\|_{L^\infty([0, T] \times \mathbb{R} \times \mathbb{R})}.$$

*Step 5: Conclusion.*

We set

$$\begin{aligned} &\mathcal{E} \left( \int_0^t F(s, X_s, (K * u^{n+1}))(s, X_s) dB_s \right) \\ &= \exp \left\{ \int_0^t F(s, X_s, (K * u^{n+1}))(s, X_s) dB_s - \frac{1}{2} \int_s^t F^2(s, X_s, (K * u^{n+1}))(s, X_s) ds \right\}, \end{aligned}$$

and

$$\mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dx dB_s \right)$$

$$\begin{aligned}
&= \exp \left\{ \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dx dB_s \right. \\
&\quad \left. - \frac{1}{2} \int_s^t \left| \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dx \right|^2 ds \right\}.
\end{aligned}$$

From inequalities in step 4 and Hölder inequality we deduce that

$$\begin{aligned}
\mathbb{E} \left[ \left| \frac{dX_{s,t}}{dx}(x) \right|^{-1} \right] &\leq \\
&\leq C \mathcal{E} \left( \int_0^t F(s, X_s, (K * u^{n+1}))(s, X_s) dB_s \right) \\
&\quad \times \mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} F_3(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dx dB_s \right)
\end{aligned} \tag{4.25}$$

Finally we observe that the processes

$$\mathcal{E} \left( \int_0^t F(s, X_s, (K * u^{n+1}))(s, X_s) dB_s \right)$$

and

$$\mathcal{E} \left( \int_0^t \int_{-\infty}^{X_s} \partial_3 F(s, x, (K * u^{n+1})) \int_{\mathbb{R}} u^{n+1}(s, y) \partial_y K(x - y) dy dx dB_s \right)$$

are martingales with expectation equal to one. From this we conclude our lemma.  $\square$

## 5 Final remarks.

**Remark 5.1.** (*Linear case*) Now suppose that  $F(t, x, z) = b(x)$ . If  $b \in L^1 \cap L^\infty$  then it satisfies the hypothesis 1. Thus, we have the uniqueness for the stochastic continuity equation with irregular drift. Then our result is the nonlinear extension of the theory of regularization by noise for transport/continuity equation initiated by Flandoli, Gubinelli and Priola in [20].

We pointed out according to the theory of Diperna-Lions (see [15]) the uniqueness of deterministic transport/continuity equation holds when  $b$  has

$W^{1,1}$  spatial regularity together with a condition of boundedness on the divergence. The theory has been generalized by L. Ambrosio [1] to the case of only BV regularity for  $b$  instead of  $W^{1,1}$ . We refer the readers to two excellent summaries in [2] and [13].

**Remark 5.2.** *The main our tool in order to have estimations on the derivative of the flow was the Itô-Wentzell-Kunita formula. However, it is possible only to apply this formula for compositions of semimartingales. In order to generalized our result for more general  $F$  we have in mind to work in the context of the theory of stochastic calculus via regularization. This calculus was introduced by by F. Russo and P. Vallois ( see [41] as general reference ) and it have been studied and developed by many authors. In the paper of F.Flandoli and F. Russo and R. Coviello and F.Russo they obtain a Itô-Wentzell-Kunita formula for more general process, see [21] and [7]. We also mention the recent extension of the Itô-Wentzell-Kunita formula by R. Duboscq and A. Reveillac in [8].*

**Remark 5.3.** *We pointed that multiplicative noise as the one used in the stochastic conservationb law (1.2) is not enough to improve the regularity of solutions of the following stochastic Burgers equation*

$$\partial_t u(t, x, ) + \partial_x u(t, x) \left( u(t, x) + \frac{dB_t}{dt} \right) = 0 .$$

*Indeed, for this equation one can observe the appearance of shocks in finite time, just as for the deterministic Burgers equation. We address the reader to [19] for a more detailed discussion of this topic.*

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